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SKEWNESS IN BANACH SPACES

BY

SIMON FITZPATRICK AND BRUCE REZNICK[1]

ABSTRACT. Let E be a Banach space. One often wants to measure how far E is from being a Hilbert space. In this paper we define the skewness $s(E)$ of a Banach space E, $0 \leqslant s(E) \leqslant 2$, which describes the asymmetry of the norm. We show that $s(E) = s(E^*)$ for all Banach spaces E. Further, $s(E) = 0$ if and only if E is a (real) Hilbert space and $s(E) = 2$ if and only if E is quadrate, so $s(E) < 2$ implies E is reflexive. We discuss the computation of $s(L^p)$ and describe its asymptotic behavior near $p = 1, 2$ and ∞. Finally, we discuss a higher-dimensional generalization of skewness which gives a characterization of smooth Banach spaces.

1. Introduction. Let E be a Banach space. One often wants to measure how far E is from being a Hilbert space. In this paper we introduce a new numerical characteristic, *skewness*, which describes the asymmetry of the norm:

$$(1.1) \quad s(E) = \sup\left\{ \lim_{t \to 0^+} \frac{\|x + ty\| - \|y + tx\|}{t} : x, y \in E, \|x\| = \|y\| = 1 \right\}.$$

It is clear from (1.1) that $0 \leqslant s(E) \leqslant 2$. We show that $s(E) = s(E^*)$ for all Banach spaces E. Further, $s(E) = 0$ if and only if E is a (real) Hilbert space and $s(E) = 2$ if and only if E is quadrate, so $s(E) < 2$ implies E is reflexive. We discuss the computation of $s(L^p)$ and describe its asymptotic behavior near $p = 1, 2$ and ∞. Finally, we discuss a higher-dimensional generalization of (1.1) which gives a characterization of smooth Banach spaces.

2. Definitions, notations and preliminaries. For $x, y \in E$ define

$$(2.1) \qquad \langle x, y \rangle = \|x\| \cdot \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t},$$

which is equal to the one-sided derivative of $\frac{1}{2}\|\cdot\|^2$ at x in the direction y. Let $[x, y] = \langle x, y \rangle - \langle y, x \rangle$; we have the following simple but useful lemma.

LEMMA 2.2. (i) $\langle x, y \rangle = \|x\| \, \|y\| \langle x/\|x\|, y/\|y\| \rangle$.
(ii) $s(E) = \sup\{[x, y] : \|x\| = \|y\| = 1\} = \sup\{[x, y] : \|x\|, \|y\| \leqslant 1\}$.
(iii) $|\langle x, y \rangle| \leqslant \|x\| \cdot \|y\|$.

PROOF. (i) This follows from $\langle ax, by \rangle = ab\langle x, y \rangle$ for $a, b \geqslant 0$.
(ii) The first equality is immediate from (1.1) and (2.1).

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For the second, note that

$$[x, y] = \|x\|\,\|y\|\left[\frac{x}{\|x\|}, \frac{y}{\|y\|}\right].$$

(iii) This is a consequence of the triangle inequality.

We now determine $s(l_p)$ for $p = 1, 2$ and ∞.

EXAMPLE 2.3. (i) $s(l_2) = 0$.

(ii) $s(l_1) = s(l_\infty) = 2$.

PROOF. (i) If $E = l_2$ then $\langle x, y\rangle$ is, in fact, the inner product so that $[x, y] \equiv 0 = s(l_2)$.

(ii) For $E = l_\infty$, fix $0 < \varepsilon < 1$ and let $x = (1, \varepsilon - 1, 0, \ldots)$ and $y = (1 - \varepsilon, 1, 0, \ldots)$. Then for t sufficiently small, $\|x + ty\| = 1 + t(1 - \varepsilon)$ and $\|y + tx\| = 1 - t(1 - \varepsilon)$. Hence $[x, y] = 2(1 - \varepsilon)$ and so $s(l_\infty) = 2$. Similarly, for $E = l_1$ let $x = (\varepsilon, 1 - \varepsilon, 0, \ldots)$ and $y = (1 - \varepsilon, -\varepsilon, 0, \ldots)$. Thus $[x, y] = 2(1 - 2\varepsilon)$ and $s(l_1) = 2$.

If the norm of E is smooth then $\langle \cdot\,, \cdot\rangle$ is the "generalized inner product" of Ritt **[R]**. Ritt showed that a smooth Banach space E satisfying $[x, y] \equiv 0$ is a Hilbert space (see Theorem 3.11 below).

3. The skewness of E and E^*. In this section we prove those properties of skewness asserted in the introduction. Recall that James **[J]** called a space *uniformly nonsquare* provided there is $\varepsilon > 0$ such that, if $\|x\| = \|y\| = 1$ and $\|x + y\| > 2 - \varepsilon$, then $\|x - y\| < 2 - \varepsilon$. We prefer to follow Day **[D]** in calling a space *quadrate* if it fails to be uniformly nonsquare. In the following theorems we exploit, sometimes implicitly, the convexity of the norm function $t \mapsto \|x + ty\|$.

THEOREM 3.1. *A Banach space E is quadrate if and only if $s(E) = 2$.*

PROOF. If $s(E) = 2$ then for $0 < \varepsilon < 1$ there exist $x, y \in E$ with $\|x\| = \|y\| = 1$ and $[x, y] > 2 - \varepsilon$. But for $0 < t < 1$,

$$\|x + y\| \geqslant \|x\| + t^{-1}(\|x + ty\| - \|x\|),$$
$$\|y - x\| \geqslant \|y\| + t^{-1}(\|y\| - \|y + tx\|).$$

Upon adding these inequalities and taking the limit as $t \to 0^+$ we obtain

$$\|x + y\| + \|y - x\| \geqslant 2 + \lim_{t \to 0^+} \frac{\|x + ty\| - \|y + tx\|}{t} > 4 - \varepsilon.$$

Since $\|x \pm y\| \leqslant 2$ we have $\|x \pm y\| > 2 - \varepsilon$; hence E is quadrate.

Conversely, suppose E is quadrate; for $0 < \varepsilon < 1$ choose $u, v \in E$ with $\|u\| = \|v\| = 1$ and $\|u \pm v\| \geqslant 2 - \varepsilon$. Let $w = u + av$ and $z = v - au$, where $0 < a < 1$ will be chosen below. By convexity, for $0 < t < a$,

$$\frac{\|w + tz\| - \|w\|}{t} \geqslant \frac{\|w\| - \|w - az\|}{a} = \frac{\|w\| - \|(1 + a^2)u\|}{a}.$$

But $\|w\| = \|u + av\| \geqslant \|u + v\| - (1 - a)\|v\| \geqslant 1 + a - \varepsilon$. Therefore,

$$\frac{\langle w, z\rangle}{\|w\|} = \lim_{t \to 0^+} \frac{\|w + tz\| - \|w\|}{t} \geqslant \frac{(1 + a - \varepsilon) - (1 + a^2)}{a} = 1 - a - \frac{\varepsilon}{a}.$$

Setting $x = w/\|w\|$ and $y = z/\|z\|$, we have by 2.2 (i),

$$\langle x, y\rangle = \frac{\langle w, z\rangle}{\|w\|\,\|z\|} \geqslant \frac{1}{\|z\|}\left(1 - a - \frac{\varepsilon}{a}\right) \geqslant \frac{1 - a - \varepsilon/a}{1 + a}.$$

Similarly,

$$\frac{\|z\| - \|z + tw\|}{t} \geqslant \frac{\|z\| - \|z + aw\|}{a} = \frac{\|z\| - \|(1 + a^2)v\|}{a}$$

yields $-\langle z, w\rangle/\|z\| \geqslant 1 - a - \varepsilon/a$ and $-\langle y, x\rangle \geqslant (1 - a - \varepsilon/a)/(1 + a)$. Taking $a = \sqrt{\varepsilon}$ we have $[x, y] = \langle x, y\rangle - \langle y, x\rangle \geqslant (2 - 4\sqrt{\varepsilon})/(1 + \sqrt{\varepsilon})$ and, since ε is arbitrary, $s(E) = 2$.

COROLLARY 3.2. *If $s(E) < 2$ then E is reflexive.*

PROOF. James [**J**] showed that a uniformly nonsquare space E (that is, an inquadrate space) must be reflexive. Indeed, E must be super-reflexive (see [**D**, p. 169]) but we will not use this fact.

To show that $s(E) = s(E^*)$ we introduce the duality map $J\colon E \to 2^{E^*}$, defined by

$$Jx = \{x^* \in E^*: x^*(x) = \|x\|^2 = \|x^*\|^2\}.$$

The following lemma is useful in the sequel. Although it is moderately well-known (see Barbu [**B**]) we include its short proof for the sake of completeness.

LEMMA 3.3. (i) *For $x, y \in E$ there exists $x^* \in Jx$ such that*

$$x^*(y) = \langle x, y\rangle = \lim_{t\to 0^+} \frac{\frac{1}{2}\|x + ty\|^2 - \frac{1}{2}\|x\|^2}{t}.$$

(ii) *If $\|x\| = 1$, $x^* \in Jx$ and $y \in E$ then $x^*(y) \leqslant \|x + y\| - \|x\|$.*

PROOF. (i) Define a linear functional g on the one-dimensional space $\mathbf{R}y$ by $g(ry) = r\langle x, y\rangle$. Since g is dominated by the sublinear functional $F(z) = \langle x, z\rangle$ we can use the Hahn-Banach theorem to get an element x^* of E^* with $x^*(y) = g(y)$ and $x^* \leqslant F$. Then $x^*(x) \leqslant F(x) = \|x\|^2$ and $-x^*(x) = x^*(-x) \leqslant -\|x\|^2$; hence $x^*(x) = \|x\|^2$. However, $x^*(z) \leqslant F(z) = \langle x, z\rangle \leqslant \|x\|\,\|z\|$ for all $z \in E$ by Lemma 2.2(iii) so $\|x^*\| \leqslant \|x\|$. Thus $\|x^*\|^2 = \|x\|^2 = x^*(x)$ and $x^* \in Jx$.

(ii) Since $x^*(x) = 1 = \|x^*\|$ we have

$$x^*(y) = x^*(x + y) - x^*(x) = x^*(x + y) - \|x\| \leqslant \|x + y\| - \|x\|.$$

THEOREM 3.4. *Let $\bar{s}(E) = \sup\{x^*(y) - y^*(x)\colon x^* \in Jx,\ y^* \in Jy,\ \|x\| = \|y\| = 1\}$; then $\bar{s}(E) = s(E)$.*

PROOF. Lemmas 2.2(ii) and 3.3(i) together show that $s(E) \leqslant \bar{s}(E)$. To get the reverse inequality, fix $0 < \varepsilon < 1$ and choose $x, y \in E$, $\|x\| = \|y\| = 1$ and $x^* \in Jx$, $y^* \in Jy$ such that $x^*(y) - y^*(x) \geqslant \bar{s}(E) - \varepsilon$. Thus for $0 < t < r < \varepsilon$ we have, by Lemma 3.3(ii) and convexity,

(3.5) $$y^*(-tx) < \|y - tx\| - \|y\| \leqslant \|y - rx\| - \|y - rx + tx\|.$$

Dividing by t and letting $t \to 0^+$ in (3.5) gives

$$(3.6)\qquad -y^*(x) \leq \lim_{t\to 0^+} \frac{\|y - rx\| - \|y - rx + tx\|}{t} = -\frac{\langle y - rx, x\rangle}{\|y - rx\|} = -\langle z, x\rangle,$$

where $z = (y - rx)/\|y - rx\|$. But by 3.3(ii) and the triangle inequality,

$$(3.7)\qquad x^*(ty) \leq \|x + ty\| - \|x\| \leq \|x + tz\| - \|x\| + t\|z - y\|.$$

Dividing by t in (3.7) and letting $t \to 0^+$ gives $x^*(y) \leq \langle x, z\rangle + \|z - y\|$. In view of (3.6),

$$(3.8)\qquad x^*(y) - y^*(x) \leq \langle x, z\rangle - \langle z, x\rangle + \|z - y\|.$$

However, $r < \varepsilon$, so $\|z - y\| \leq 2\varepsilon/(1 - \varepsilon)$. From (3.8) we see that $[x, z] \geq \bar{s}(E) - \varepsilon - 2\varepsilon/(1 - \varepsilon)$, and since ε is arbitrary and $\|x\| = \|z\| = 1$, we conclude that $s(E) \geq \bar{s}(E)$.

COROLLARY 3.9. *For any Banach space E, $s(E) = s(E^*)$.*

PROOF. If $s(E) = 2$ then E is quadrate, which implies by **[J]** that E^* is quadrate, so $s(E^*) = 2$. If $s(E) < 2$ then E is reflexive. By the definitions of J and $\bar{s}(E)$ we may rewrite $\bar{s}(E)$ as

$$(3.10)\qquad \bar{s}(E) = \sup\{x^*(y) - y^*(x) \colon x, y \in E, x^*, y^* \in E^*, \\ x^*(x) = y^*(y) = \|x\| = \|y\| = \|x^*\| = \|y^*\| = 1\}.$$

Since (3.10) is symmetric in E and E^* and $E^{**} = E$, $\bar{s}(E) = \bar{s}(E^*)$. By Theorem 3.4, $s(E) = s(E^*)$.

THEOREM 3.11. *If E is a Banach space and $s(E) = 0$ then E is a Hilbert space with inner product $\langle \cdot, \cdot\rangle$ and $\langle x, x\rangle = \|x\|^2$.*

PROOF. By Theorem 3.4, $\bar{s}(E) = 0$; it follows easily that for all $x, y \in E$ and $x^* \in Jx$, $y^* \in Jy$, we have $x^*(y) = y^*(x)$. Now pick $x \in E$ and $x^* \in Jx$. For any $y \in E$ there exists $y^* \in Jy$ so that $\langle y, x\rangle = y^*(x)$ (by 3.3(i)); hence $\langle y, x\rangle = x^*(y)$ is linear in y and, similarly, linear in x. Finally, $\langle x, x\rangle = \|x\|^2$ immediately from the definition. (Alternatively, one can show that $s(E) = 0$ implies E is smooth, and apply the main theorem of **[R]**.)

REMARK. Recall that, in Lemma 2.3(ii) and Theorem 3.1, we proved that $s(E) = 2$ by exhibiting x and y with $[x, y] \geq 2 - \varepsilon$. It is natural to ask whether there exist a Banach space E and $x, y \in E$ with $\|x\| = \|y\| = 1$ and $[x, y] = 2$.

Suppose $[x, y] = 2$. Necessarily $\langle x, y\rangle = 1$ and by Lemma 3.3, there is $x^* \in Jx$ with $x^*(y) = \langle x, y\rangle = 1$, while $x^*(x) = 1 = \|x^*\|^2$. Thus $1 + t \geq \|x + ty\| \geq x^*(x + ty) = 1 + t$ for $t \geq 0$. Therefore $\|y + tx\| = t\|x + t^{-1}y\| = t(1 + t^{-1}) = 1 + t$, so $[x, y] = 0$, a contradiction.

4. Skewness of L^p. We have already seen that $s(L^1) = s(L^\infty) = 2$, $s(L^2) = 0$ and $s(L^p) = s(L^q)$ for conjugate p and q. In this section, we compute $s(p) := s(L^p)$ for $2 < p < \infty$ indirectly (of course, $s(p) = s(p/(p - 1))$ for $1 < p < 2$), we give some particular values and discuss, without detailed proofs, the asymptotics of $s(p)$. Our major result is Theorem 4.1: The proof applies to all $L^p(X, \Sigma, \mu)$ provided there are two disjoint elements of Σ with positive mass.

THEOREM 4.1. *For* $2 < p < \infty$, $s(p) = \max_{t>0} 2(t - t^{p-1})/(1 + t^p)$. *This maximum is achieved at* $t = t_p$, *where* t_p *is the unique solution*, $0 < t_p < 1$, *of* $t^{p-1} + t^{1-p} = (p-1)(t + t^{-1})$.

PROOF. Pick $f, g \in L^p$ with $\|f\| = \|g\| = 1$. By Lemma 3.3(i), there exist $f^* \in Jf$ and $g^* \in Jg$ so that $[f, g] = f^*(g) - g^*(f)$, but by Hölder's inequality, $h^* \in Jh$ with $\|h^*\| = \|h\| = 1$ implies $h^*(\omega) = |h(\omega)|^{p-1}\operatorname{sgn} h(\omega)$ μ-a.e. Accordingly,

$$[f, g] = \int \left(|f|^{p-1}|g| - |f||g|^{p-1}\right)\operatorname{sgn} f \operatorname{sgn} g \, d\mu. \tag{4.2}$$

(This formula can also be found by differentiating under the integral sign.)

Now suppose that λ is chosen so that

$$\left(|u|^{p-1}|v| - |u||v|^{p-1}\right)\operatorname{sgn} u \operatorname{sgn} v \leqslant \lambda\left(|u|^p + |v|^p\right) \tag{4.3}$$

for all real u, v. Then by integrating (4.3) pointwise with $u = f(\omega)$, $v = g(\omega)$, $[f, g] \leqslant 2\lambda$. In order to determine which λ satisfy (4.3) for all real u, v, we may assume that $u > 0$. If $v > 0$ then dividing by u^p gives $\lambda \geqslant (t - t^{p-1})/(1 + t^p)$ for $t = v/u$. If $v < 0$, the same inequality arises from dividing by $|v|^p$ and letting $t = -u/v$. Accordingly, (4.3) holds when $\lambda = \lambda_p = \max(t - t^{p-1})/(1 + t^p)$. Conversely, let A and B be two sets of positive mass, let $g \equiv t_p f$ on A and $f \equiv -t_p g$ on B, $g > 0$ chosen so that $\int_A f^p \, d\mu = \int_B g^p \, d\mu = (t_p^p + 1)^{-1}$. Then (4.3) is an equality and $\|f\| = \|g\| = 1$ with $[f, g] = 2\lambda_p$.

We now present some properties of the function $s(p)$.

THEOREM 4.4. *For* $2 < p < \infty$, $s(p)$ *is monotonically increasing*, $s(4) = 1/\sqrt{2}$ *and* $s(6) = 1$.

PROOF. Let $w(t, p) = 2(t - t^{p-1})/(1 + t^p)$. Then

$$\frac{\partial w}{\partial p} = -2 \log t \frac{t^{p-1} + t^{p+1}}{(1 + t^p)^2} > 0,$$

for $0 < t < 1$. Thus, for $q > p$, $s(q) = w(t_q, q) \geqslant w(t_p, q) > w(t_p, p) = s(p)$. For $p = 4$, t_4 is the root of $t^3 + t^{-3} = 3(t + t^{-1})$. Letting $u = t_4 + t_4^{-1}$, $u^3 - 3u = 3u$, so $u = 0, \pm\sqrt{6}$. We thus have $t_4 + t_4^{-1} = \sqrt{6}$, or $t_4 = (\sqrt{6} - \sqrt{2})/2$; it turns out that $w(t_4, 4) = 1/\sqrt{2}$. Similarly, t_6 is the root of $t^5 + t^{-5} = 5(t + t^{-1})$ and, for $u = t_6 + t_6^{-1}$, $u^5 - 5u^3 = 0$, so $t_6 = (\sqrt{5} - 1)/2$ and $w(t_6, 6) = 1$.

In general, we cannot hope for an explicit formula for $s(p)$. Indeed the only other tractable value is for $p = 3$: $t_3 = (1 + \sqrt{3} - 12^{1/4})/2$, $s(3) = \sqrt{2}(\sqrt{3} - 1)3^{-3/4}$. Some numerical computations made on an HP-41C are shown in Table 4.5.

p	$s(p)$	p	$s(p)$	p	$s(p)$
1	2.0000	2	.0000	4	.7071
1.001	1.9829	2.001	.0007	5	.8763
1.01	1.8779	2.01	.0066	6	1.0000
1.05	1.5803	2.1	.0632	7	1.0955
1.1	1.3330	2.2	.1207	8	1.1720
1.2	1.0000	2.3	.1736	9	1.2349
1.3	.7702	2.4	.2224	10	1.2878
1.4	.5951	2.5	.2677	20	1.5649
1.5	.4542	2.6	.3099	40	1.7430
1.6	.3365	2.7	.3493	100	1.8769
1.7	.2357	2.8	.3864	1000	1.9829
1.8	.1477	2.9	.4213	10000	1.9978
1.9	.0698	3	.4542	∞	2

TABLE 4.5

The asymptotic analysis of $s(p)$ is based on the following observations. Suppose $t^r + t^{-r} = r(t + t^{-1})$, $(r = p - 1)$; if $t = e^z$ then $\cosh rz = r \cosh z$. Letting $q(u) = u^{-1}\cosh(u)$, we see that $q(rz) = q(z)$. We now look at the equation $q(u) = q(v)$ with $v/u = r$. We state, without proof or error estimates, the following theorem.

THEOREM 4.6. *Asymptotically,* $s(1+\varepsilon) \simeq 2 + 2\varepsilon\log\varepsilon$, $s(2+\varepsilon) \simeq \alpha|\varepsilon|$ *and* $s(p) \simeq 2 - \alpha\log p/p$ *for large* p. *Here,* $\alpha = 2y_0/(y_0^2 - 1)$, *where* y_0 *is the root of* $(\log y)(y^2 - 1) = y^2 + 1$; $y_0 \simeq 3.319$, $\alpha \simeq .663$.

5. Generalizations of skewness. In this section we generalize skewness to allow for an arbitrary (finite) number of points and derive a family of characterizations of smooth Banach spaces. We wish to thank Professor A. Pełczynski for suggesting to us a generalization of this kind.

For $n \geq 2$ and $x_i \in E$, define the following expression:

$$(5.1)\qquad [x_1,\dots,x_n]_n = \limsup_{\underline{t}\to\underline{0}} \frac{\sum \operatorname{sgn}(\sigma)\|x_{\sigma(1)} + t_2x_{\sigma(2)} + \cdots + t_nx_{\sigma(n)}\|}{|t_2| + \cdots + |t_n|},$$

where the summation in (5.1) is taken over all permutations $\sigma \in S_n$. Note that for $n = 2$ we have $[x, y]_2 = \max([x, y], [x, -y])$. We now define n-dimensional skewness:

$$(5.2)\qquad s_n(E) = \sup\{[x_1,\dots,x_n]_n : \|x_1\| = \cdots = \|x_n\| = 1\};$$

observe that $s_2(E) = s(E)$. Since n is clear from the context, we drop the outer subscript on $[x_1,\dots,x_n]$ for $n \geq 3$. We will establish a peculiar branching in the behavior of s_n: s_3 is much like s_2, but s_n for $n \geq 4$ selects out nonsmoothness in Banach spaces.

THEOREM 5.3. *If* E *is a smooth Banach space then* $2s(E) \leq s_3(E) \leq 3s(E)$.

PROOF. Letting u^* denote the derivative of the norm at u, a routine computation shows that

(5.4)

$$[x, y, z] = \limsup_{(t_2, t_3) \to (0,0)} \left\{ \frac{t_2 - t_3}{|t_2| + |t_3|} (x^*(y - z) + y^*(z - x) + z^*(x - y)) \right\}.$$

By Theorem 3.4, $x^*(y) - y^*(x) \leqslant s(E)$, etc., so $[x, y, z] \leqslant 3s(E)$. On the other hand, pick $z = -y$ and take $t_2 > 0$, $t_3 = 0$ in (5.4). Since $(-y)^* = -y^*$, $[x, y, -y] \geqslant x^*(2y) - 2y^*(x) = 2(x^*(y) - y^*(x))$. If we now take the supremum over $\|x\| = \|y\| = 1$, we get $s_3(E) \geqslant 2s(E)$.

A more delicate analysis is necessary when E is not smooth.

LEMMA 5.5. *Suppose $x, y \in E$ and α, β are real, with $\|x\| = 1$. Then*

$$\|(1 + \alpha t)x + \beta t y\| = \|x + \beta t y\| + \alpha t + o(t)$$

for small t.

PROOF. We have $\|(1 + \alpha t)x + \beta t y\| = (1 + \alpha t)\|x + \beta t(1 + \alpha t)^{-1} y\| = (1 + \alpha t)(\|x + \beta t y\| + o(t)) = \|x + \beta t y\| + \alpha t\|x\| + o(t)$.

THEOREM 5.6. *For any Banach space E, $s_3(E) \geqslant 2s(E)$.*

PROOF. As in the proof of Theorem 5.3, we compute $[u, v, -v]$ for appropriate $u, v \in E$. Taking $t_2 = t > 0$ and $t_3 = 0$ in (5.1), we have, after a rearrangement of terms,

$$(5.7) \quad [u, v, -v] \geqslant \lim_{t \to 0^+} \left(\frac{\|u + tv\| - \|v + tu\|}{t} + \frac{\|u - tv\| - \|v - tu\|}{-t} \right).$$

Thus it suffices to find u, v so that $\|u\| = \|v\| = 1$ and

$$(5.8) \qquad \liminf_{t \to 0} \frac{\|u + tv\| - \|v + tu\|}{t} \geqslant s(E) - 2\varepsilon.$$

Pick x, y with $\|x\| = \|y\| = 1$ and $[x, y] \geqslant s(E) - \varepsilon$. For any $\alpha > 0$, the convexity of the norm implies

$$(5.9) \qquad \liminf_{t \to 0} \frac{\|x + \alpha y + ty\| - \|x + \alpha y\|}{t} \geqslant \langle x, y \rangle,$$

$$\limsup_{t \to 0} \frac{\|y - \alpha x + ty\| - \|y - \alpha x\|}{t} \leqslant \langle y, x \rangle.$$

Let $u = (x + \alpha y)/\|x + \alpha y\|$ and $v = (y - \alpha x)/\|y - \alpha x\|$, where $0 < \alpha < \varepsilon/5$, so $\|u - x\|, \|v - y\| \leqslant 2\alpha/(1 - \alpha) < \varepsilon/2$. Then

$$(5.10) \quad \liminf_{t \to 0} \frac{\|u + tv\| - 1}{t} = \liminf_{t \to 0} \frac{\|x + \alpha y + t\|x + \alpha y\| y\| - \|x + \alpha y\|}{t\|x + \alpha y\|}$$

$$= \liminf_{t \to 0} \frac{\|x + \alpha y + ty\| - \|x + \alpha y\|}{t} \geqslant \langle x, y \rangle.$$

Similarly, $\limsup_{t\to 0} t^{-1}(\|v + tx\| - 1) \leqslant \langle y, x\rangle$ so that

$$(5.11)\qquad \liminf_{t\to 0} \frac{\|u + ty\| - \|v + tx\|}{t} \geqslant \langle x, y\rangle - \langle y, x\rangle \geqslant s(E) - \varepsilon.$$

Since $\|u - x\|$, $\|v - y\| \leqslant \varepsilon/2$, (5.11) implies (5.8) and we are done.

COROLLARY 5.12. *If $s_3(E) = 0$ then E is a Hilbert space.*

THEOREM 5.13. *For any Banach space E, $s_3(E) \leqslant 3s(E)$.*

PROOF. For $x, y \in E$, let $z = x/\|x\|$. Then by Lemma 3.3(ii), $\|x\| - \|x + y\| = \|x\|(\|z\| - \|z + y/\|x\|\|) \leqslant -\|x\|z^*(y/\|x\|) = -z^*(y) = z^*(-y)$ for $z^* \in Jz$. Thus, for any $u, v, w \in E$,

$$(5.14)\qquad \|u + t_2v + t_3w\| - \|u + t_2w + t_3v\| \leqslant u^*((t_2 - t_3)(v - w)),$$

for $u^* \in J((u + t_2v + t_3w)/\|u + t_2v + t_3w\|)$. For $(u, v, w) = (x_1, x_2, x_3)$, (x_2, x_3, x_1), (x_3, x_1, x_2), let u^* be denoted by x_1^*, x_2^*, x_3^* respectively. These functionals depend on t_2 and t_3 of course. Then

(5.15)

$$[x_1, x_2, x_3] \leqslant \limsup_{\underline{t}\to\underline{0}} \frac{t_2 - t_3}{|t_2| + |t_3|}\{x_1^*(x_2 - x_3) + x_2^*(x_3 - x_1) + x_3^*(x_1 - x_2)\}.$$

As $(t_2, t_3) \to (0, 0)$ we can choose weak* cluster points y_i^* for x_i^* since the unit ball is weak* compact in E^*. It is easily checked that $y_i^* \in J(x_i)$ and, since $t_2 - t_3 \leqslant |t_2| + |t_3|$, $[x_1, x_2, x_3] \leqslant y_1^*(x_2 - x_3) + y_2^*(x_3 - x_1) + y_3^*(x_1 - x_2)$. As in the proof of Theorem 5.3 this implies $s(E) \leqslant 3s(E)$.

When $n \geqslant 4$ there is again a dichotomy between smooth and nonsmooth Banach spaces. The differences, however, are now irreconcilable.

THEOREM 5.16. *For some i, $1 \leqslant i \leqslant n$, let x_i be a point of smoothness for the norm of E and suppose $n \geqslant 4$. Then*

$$(5.17)\qquad \limsup_{\underline{t}\to\underline{0}} \frac{\sum'\operatorname{sgn}(\sigma)\|x_{\sigma(1)} + t_2x_{\sigma(2)} + \cdots + t_nx_{\sigma(n)}\|}{|t_2| + \cdots + |t_n|} = 0$$

where the summation in (5.17) *is taken over all $\sigma \in S_n$ with $\sigma(1) = i$.*

PROOF. Let x_i^* denote the derivative of the norm at x_i; then we can rewrite (5.17) as (5.18)

$$(5.18)\qquad \limsup_{\underline{t}\to\underline{0}} \frac{\sum'\operatorname{sgn}(\sigma)\left(\|x_i\| + \sum_{j=2}^{n} t_jx_i^*(x_{\sigma(j)}) + o(t)\right)}{|t_2| + \cdots + |t_n|},$$

or, reversing the order of summation,

$$\limsup_{\underline{t}\to\underline{0}} \frac{\sum_{j=2}^{n}\sum_{k=1}^{n}\left(\sum_{j,k}'\operatorname{sgn}(\sigma)\right)\left(t_jx_i^*(x_k) + \|x_i\|\right)}{|t_2| + \cdots + |t_n|},$$

where $\Sigma'_{j,k}$ denotes the sum over all permutations σ for which $\sigma(1) = i$, $\sigma(j) = k$. Since $n \geqslant 4$, an equal number of such permutations are even and odd; thus the inner sum vanishes and the theorem is proved.

COROLLARY 5.19. *For any smooth Banach space E, $s_n(E) = 0$ for $n \geqslant 4$.*

When $n \geqslant 4$, smoothness is also necessary for $s_n(E) = 0$, as we now show.

THEOREM 5.20. *If E is not smooth then $s_n(E) > 0$ for every n.*

PROOF. If $n = 2, 3$, only Hilbert spaces have $s_n(E) = 0$. Let $n \geqslant 4$ and let x be a point of nonsmoothness in the direction y, $\|x\| = \|y\| = 1$ and, without loss of generality, suppose that E is spanned by x and y (this can only decrease $s_n(E)$). Since E is two-dimensional, it follows from (5.5) that the norm is not smooth at x in any direction of E other than $\pm x$. On the other hand, the function $t \mapsto \|x + ty\|$ is convex and so is differentiable except at a countable number of points. By considering $y' = rx + sy$, $\|y'\| = 1$ if necessary, we can assume that the norm is smooth at y; further, we can select a large number of elements $\alpha x + \beta y$, $\|\alpha x + \beta y\| = 1$, at which the norm is smooth.

Let d_+ and d_- denote $\lim_{t \to 0^\pm} t^{-1}(\|x + ty\| - \|x\|)$; without loss of generality, suppose $d_+ > d_-$. We now let $x_1 = x$, $x_2 = -y$ and $x_i = \alpha_i x + \beta_i y$ for $3 \leqslant i \leqslant n$; where $\|x_i\| = 1$, the norm is smooth at x_i, $i \geqslant 2$, and $(n-2)^{-1} > \beta_3 > \cdots > \beta_n > 0$. Choose $\lambda_2, \ldots, \lambda_n$ with $\lambda_2 = 1$ and $\lambda_3 > \cdots > \lambda_n > 1$ so that $\Sigma_{i=3}^n \beta_i \lambda_i > 1 > \Sigma_{i=3}^n \beta_i \lambda_{\pi(i)}$ for any permutation π of $\{2, \ldots, n\}$ other than the identity. Consider (5.1) with $t_i = \lambda_i t$, $t > 0$, and let $\lambda = \Sigma_{i=2}^n \lambda_i$. We have

$$(5.21) \quad \lambda[x_1, \ldots, x_n] \geqslant \lim_{t \to 0^+} t^{-1} \sum{}' \operatorname{sgn}(\sigma) \|x + tx_{\sigma(2)} + \lambda_3 t x_{\sigma(3)} + \cdots + \lambda_n t x_{\sigma(n)}\|,$$

where Σ' denotes the sum over permutations with $\sigma(1) = 1$; the contribution from the other σ's vanishes by Theorem 5.16. We might as well sum in (5.21) over σ^{-1}; hence,

(5.22)

$$\begin{aligned} \lambda[x_1, \ldots, x_n] &\geqslant \lim_{t \to 0^+} t^{-1} \sum{}' \operatorname{sgn}(\sigma) \left\| x + t\lambda_{\sigma(2)}(-y) + \sum_{i=3}^n t\lambda_{\sigma(i)}(\alpha_i x + \beta_i y) \right\| \\ &= \lim_{t \to 0^+} t^{-1} \sum{}' \operatorname{sgn}(\sigma) \left\| \left(1 + \sum_{i=3}^n t\lambda_{\sigma(i)}\alpha_i\right) x + \left(\sum_{i=3}^n \beta_i \lambda_{\sigma(i)} - \lambda_{\sigma(2)}\right) ty \right\|. \end{aligned}$$

Apply Lemma 5.5 to (5.22) and note that $\Sigma' \operatorname{sgn}(\sigma) \Sigma_{i=3}^n \lambda_{\sigma(i)} \alpha_i$ vanishes; thus,

$$(5.23) \quad \lambda[x_1, \ldots, x_n] \geqslant \lim_{t \to 0^+} t^{-1} \sum{}' \operatorname{sgn}(\sigma) \left\| x + \left(\sum_{i=3}^n \beta_i \lambda_{\sigma(i)} - \lambda_{\sigma(2)}\right) ty \right\|.$$

By choice, $\Sigma_{i=3}^n \beta_i \lambda_i - 1$ is positive, but for other σ's, $\Sigma_{i=3}^n \beta_i \lambda_{\sigma(i)} - \lambda_{\sigma(2)}$ is negative. Recalling the definition of $d_\pm$, we have

(5.24)

$$\lambda[x_1,\ldots,x_n] = \left(\sum_{i=3}^n \beta_i\lambda_i - 1\right)d_+ + \left(\sum_{\sigma\neq \mathrm{id}} \mathrm{sgn}(\sigma)\left(\sum_{i=3}^n \beta_i\lambda_{\sigma(i)} - \lambda_{\sigma(2)}\right)\right)d_-$$
$$= \left(\sum_{i=3}^n \beta_i\lambda_i - 1\right)(d_+ - d_-) > 0.$$

We conclude with some computations.

THEOREM 5.25. (i) *For* $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$, $s_3(L^p) = s_3(L^q)$.

(ii) *For* $1 < p < \infty$, $s_3(L_p) = 3\sup \Phi(r, s, t)(|r|^p + |s|^p + |t|^p)^{-1}$, *where*

$$\Phi(r,s,t) = \left(|r|^{p-1}|s| - |r||s|^{p-1}\right)\mathrm{sgn}(rs) + \left(|s|^{p-1}|t| - |s||t|^{p-1}\right)\mathrm{sgn}(st) + \left(|t|^{p-1}|r| - |t||r|^{p-1}\right)\mathrm{sgn}(rt).$$

PROOF. (i) Since L^p is smooth, by (5.4) and the proof of Theorem 4.1, $[f, g, h] = \int \Phi(f, g, h)\, d\mu$. For $k \in L^p$, let $k^*(\omega) = |k(\omega)|^{p-1}\mathrm{sgn}\, k(\omega)$. Then $k^* \in L^q$ $(p-1 = p/q)$, $\|k^*\| = 1$ and it is easily checked that $[f, g, h] = [f^*, h^*, g^*]$ so that $s_3(L^p) \leqslant s_3(L^q) \leqslant s_3(L^p)$.

(ii) Let λ_p denote the supremum of $\Phi(r, s, t)(|r|^p + |s|^p + |t|^p)^{-1}$. Then, as in the proof of Theorem 4.1, $s_3(L^p) \leqslant 3\lambda_p$ and the maximum is achieved since we may assume it to occur on $|r|^p + |s|^p + |t|^p = 1$. Provided (X, μ) has three disjoint sets of positive mass, we can construct f, g, h for which $[f, g, h] = 3\lambda_p$, as before.

We remark that it seems very hard to compute $s_3(L^p)$. The only nontrivial explicit value we know is $s_3(L_4) = (3/2)^{7/4} \simeq 2.0331$. The computation is very indirect, following from representation of

$$r^4 + s^4 + t^4 - 2^{7/4}3^{-3/4}\left(r^3(s-t) + s^3(t-r) + t^3(r-s)\right)$$

as a sum of squares of quadratic forms.

EXAMPLE 5.26. (i) $s_3(l_1) = s_3(l_\infty) = 6$.

(ii) $s_4(l_\infty) \geqslant 6$.

PROOF. (i) For any E, $s_3(E) \leqslant 3s(E) \leqslant 6$. On the other hand, take $\underline{t} = (t, 0)$ and points $(1-2\varepsilon, \varepsilon, -\varepsilon)$, $(-\varepsilon, 1-2\varepsilon, \varepsilon)$, $(\varepsilon, -\varepsilon, 1-2\varepsilon)$ for l_1 and $(1, \varepsilon - 1, 1-\varepsilon)$, $(1-\varepsilon, 1, \varepsilon - 1)$, $(\varepsilon - 1, 1-\varepsilon, 1)$ for l_∞, where $\varepsilon > 0$ is small. The calculations resemble (2.3).

(ii) Take $\underline{t} = (t, 0)$ and $x_i \in l_\infty^{12}$, where $x_1 = (A, B, C, D)$, $x_2 = (B, A, D, C)$, $x_3 = (C, D, A, B)$, $x_4 = (D, C, B, A)$, $A = (1, 1, 1)$, $B = (\alpha, 0, -\alpha)$, $C = (-\alpha, \alpha, 0)$, $D = (0, -\alpha, \alpha)$ and $\alpha = 1 - \varepsilon$.

Questions 5.27. Is $s_3(E) = s_3(E^*)$? (There are well-known smooth E with non-smooth E^* and vice versa.) What are $s_n(l_1)$, $s_n(l_\infty)$ for $n \geqslant 4$? What is the behavior of $n \mapsto s_n(E)$ for $n \geqslant 2$? What is $\sup_E s_n(E)$? For which spaces is it achieved?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, ILLINOIS 61801 (Current address of Bruce Reznick)

Current address (Simon Fitzpatrick): Department of Mathematics, University of Auckland, Private Bag, Auckland, New Zealand